Mathematical Foundation of Fluid Mechanics

Outline of the lecture

- Conservation of transportable quantities
- The general form of the equations
- Parabolic-elliptic and hyperbolic equations
- The choice of dependent and independent variables
- Boundary conditions
- The treatment of complex geometry
- The first steps

Conservation laws

Conservation of mass
Conservation of momentum
Conservation of energy (1$_{st}$ law of Thermodynamics)
Conservation of species
Second Law of Thermodynamics

•Each of the above equations has one transportable quantity as dependent variable and

•every equation expresses the balance of various transport mechanisms

Balance of transport mechanisms

The temporal rate of change of a transportable quantity per volume

=

transport through convection

+

transport through diffusion

+

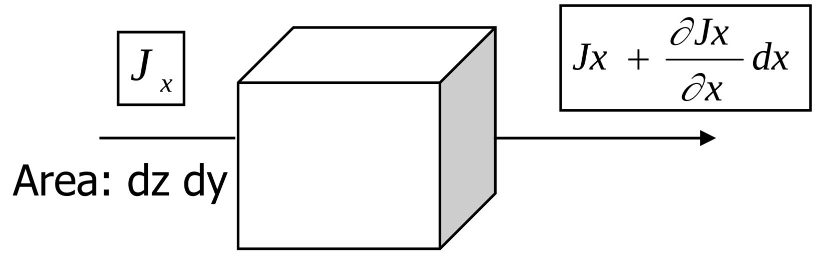
production or destruction of the quantity

Convective transport of variable

$$\vec{J} = \rho \vec{V} \Phi$$

Components J_x, J_y, J_z

$J_x = \rho u \Phi$, $J_y = \rho v \Phi$, $J_z = \rho w \Phi$



$$\{(J_x + dJ_x)dy.dz - J_x dy.dz\} / dx.dy.dz$$

- **Balance of convective transport**

$$\boxed{div\,\vec{J} = \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} \quad \text{per} \quad \text{unit} \quad \text{volume}}$$

Diffussive transport/1

Basic Diffusion laws:

•Newton's law for "diffusion of momentum"

•Fourier's law for "diffusion of heat"

•Fick's law for "diffusion of species"

$$\tau_{ij} = \mu\left(\frac{\vartheta u_i}{\vartheta x_j} + \frac{\vartheta u_j}{\vartheta u_i}\right) , \; q = -k\nabla T \; , \; q_c = -\frac{k}{Sc}\nabla C$$

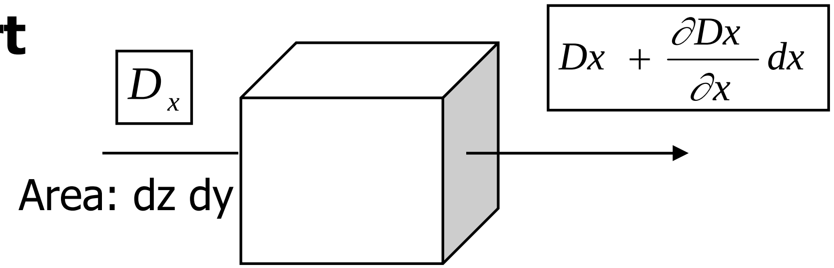
All Diffusion laws are of the gradient type

$$\vec{D} = -\Gamma_{\varphi}\, grad\ \Phi$$

Diffussive transport/2

Balance of diffusive transport

$$div\,\vec{D} = -div\left(\Gamma_\phi\, grad\,\Phi\right)$$



Combined convective and diffusive transport

$$div\left(\vec{J} + \vec{D}\right) = div\left(\rho\vec{V}\Phi - \Gamma_\phi\, grad\,\Phi\right)$$

$$= div(\rho\vec{V}\,\Phi) - div(\Gamma_\phi\, grad\Phi)$$

Production and destruction terms

examples of such terms S_Φ

- **-grad p**

Body forces as

- **Coriolis forces in atmospheric flows,**
- **buoyant forces,**
- **centrifugal forces,**
- **electrostatic forces**
- **volume or area sources of heat**
- **Kinetic heating**
- **phase changes**
- **chemical reactions**

General form of the balance equation

$$\frac{\partial(\rho\Phi)}{\partial t} + \mathrm{div}\left(\rho\vec{V}\Phi - \Gamma_{\Phi}\,\mathrm{grad}\,\Phi\right) = S_{\Phi}$$

net rate of change **convection** **diffusion** **source**

- **div and grad expressed on various coordinate systems i.e. Cartesian, Curvilinear orthogonal, curvilinear, polar**

Choice of dependent variables & coordinate system

u,v,w,p,T
Ω,ψ (2D)

Cartesian velocities & Cartesian coordinates

Contravariant velocities & Curvilinear

Orthogonal or not coordinates

Cartesian velocities & Curvilinear coordinates

Solution in physical or transformed space

Need for creating curvilinear body fitted grid

Εξισώσεις διατήρησης μάζας (καρτεσιανό σύστημα συντεταγμένων)

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \vec{V} = 0$$ **(μη συντηρητική μορφή)**

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \vec{V} \right) = 0$$ **(συντηρητική μορφή)**

Νόμοι διατήρησης ορμής (μη συντηρητική έκφραση)

$$\rho \frac{Du}{Dt} = -\frac{\partial P}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho f_x$$ x-συνιστώσα ορμής

$$\rho \frac{Dv}{Dt} = -\frac{\partial P}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho f_y$$ y-συνιστώσα ορμής

$$\rho \frac{Dw}{Dt} = -\frac{\partial P}{\partial z} + \underbrace{\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}}_{\text{Διατμητικές τάσεις}} + \rho f_z$$ z-συνιστώσα ορμής

Ολική παράγωγος

Κλίση πίεσης

Διατμητικές τάσεις

Εξωτερικές δυνάμεις

Εξισώσεις ορμής (συντηρητική έκφραση)

$$\frac{\partial(\rho u)}{\partial t}+\nabla\cdot\left(\rho u\vec{V}\right)=-\frac{\partial P}{\partial x}+\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}+\rho f_x$$ x-συνιστώσα ορμής

$$\frac{\partial(\rho v)}{\partial t}+\nabla\cdot\left(\rho v\vec{V}\right)=-\frac{\partial P}{\partial y}+\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y}+\frac{\partial \tau_{zy}}{\partial z}+\rho f_y$$ y-συνιστώσα ορμής

$$\frac{\partial(\rho w)}{\partial t}+\nabla\cdot\left(\rho w\vec{V}\right)=-\frac{\partial P}{\partial z}+\frac{\partial \tau_{zy}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \tau_{zz}}{\partial z}+\rho f_z$$ z-συνιστώσα ορμής

Εξίσωση διατήρησης ενέργειας (μη συντηρητική έκφραση)

$$\rho \frac{D}{Dt}\left(e+\frac{V^2}{2}\right)=\rho\,\dot{q}+\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)$$

$$-\frac{\partial(uP)}{\partial x}-\frac{\partial(vP)}{\partial y}-\frac{\partial(wP)}{\partial z}+\frac{\partial(u\tau_{xx})}{\partial x}+\frac{\partial(u\tau_{yx})}{\partial y}+$$

$$+\frac{\partial(u\tau_{zx})}{\partial z}+\frac{\partial(v\tau_{xy})}{\partial x}+\frac{\partial(v\tau_{yy})}{\partial y}+\frac{\partial(v\tau_{zy})}{\partial z}+$$

$$+\frac{\partial(w\tau_{xz})}{\partial x}+\frac{\partial(w\tau_{yz})}{\partial y}+\frac{\partial(w\tau_{zz})}{\partial z}+\rho\,\vec{f}\cdot\vec{V}$$

$$\rho \frac{D}{Dt}\left(e+\frac{V^2}{2}\right)=\frac{\partial}{\partial t}\left[\rho\left(e+\frac{V^2}{2}\right)\right]+\nabla\cdot\left[\rho\left(e+\frac{V^2}{2}\right)\vec{V}\right]$$

(συντηρητική έκφραση)

Ορθές διατμητικές τάσεις

$$\tau_{xx} = \frac{2\mu}{3}(2\frac{\vartheta u}{\vartheta x} - \frac{\vartheta v}{\vartheta y} - \frac{\vartheta w}{\vartheta z}) \quad \tau_{yy} = \frac{2\mu}{3}(2\frac{\vartheta v}{\vartheta y} - \frac{\vartheta u}{\vartheta x} - \frac{\vartheta w}{\vartheta z}) \quad \tau_{zz} = \frac{2\mu}{3}(2\frac{\vartheta w}{\vartheta z} - \frac{\vartheta u}{\vartheta x} - \frac{\vartheta v}{\vartheta y})$$

$$\tau_{xy} = \mu(\frac{\vartheta u}{\vartheta y} + \frac{\vartheta v}{\vartheta x}) \quad \tau_{yz} = \mu(\frac{\vartheta u}{\vartheta z} + \frac{\vartheta w}{\vartheta y}) \quad \tau_{xz} = \mu(\frac{\vartheta u}{\vartheta z} + \frac{\vartheta w}{\vartheta x})$$

Εξίσωση μεταφοράς εσωτερικής ενέργειας

Προκύπτει από την εξίσωση διατήρησης της ενέργειας

$$\rho \frac{De}{Dt} = \rho\,\dot{q} + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)$$

$$- P\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \tau_{xx}\frac{\partial u}{\partial x} + \tau_{yx}\frac{\partial u}{\partial y} + \tau_{zx}\frac{\partial u}{\partial z}$$

$$+ \tau_{xy}\frac{\partial v}{\partial x} + \tau_{yy}\frac{\partial v}{\partial y} + \tau_{zy}\frac{\partial v}{\partial z}$$

$$+ \tau_{xz}\frac{\partial w}{\partial x} + \tau_{yz}\frac{\partial w}{\partial y} + \tau_{zz}\frac{\partial w}{\partial z}$$

$$\rho \frac{De}{Dt} = \rho\,\dot{q} + \nabla\cdot\left(k\nabla T\right) - P\left(\nabla\cdot\vec{V}\right) + \mu\,\Phi$$

$$\Phi = \left[2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2\right] -$$

$$-\frac{2}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)^2$$

Φ συνάρτηση απορρόφησης)

Εξίσωση μεταφοράς εντροπίας

$$d\,e = T\,d\,S - p\,d\left(\frac{1}{\rho}\right) \qquad \rho\frac{Ds}{Dt} = \frac{\rho}{T}\frac{De}{Dt} - \frac{P}{\rho T}\frac{D\rho}{Dt}$$

$$\rho\frac{Ds}{Dt} = \frac{1}{T}\nabla\cdot\left(k\,\nabla\,T\right) + \frac{\mu}{T}\Phi$$

$$\rho\frac{Ds}{Dt} = \nabla\left(\frac{k}{T}\nabla\,T\right) + S'''_{gen}$$

$$S'''_{gen} = \frac{k}{T^2}\left[\left(\frac{\partial T}{\partial x}\right)^2 + \left(\frac{\partial T}{\partial y}\right)^2 + \left(\frac{\partial T}{\partial z}\right)^2\right] + \frac{\mu}{T}\Phi\left(\frac{watt}{m^3}K\right)$$

Μητρωική μορφή εξισώσεων διατήρησης

$$\frac{\partial U}{\partial t}+\frac{\partial F}{\partial x}+\frac{\partial G}{\partial y}+\frac{\partial H}{\partial z}=J$$

$$U=\begin{Bmatrix} \rho \\ \rho\, u \\ \rho\, v \\ \rho\, w \\ \rho\left(e+\dfrac{V^2}{2}\right) \end{Bmatrix}, \qquad F=\begin{Bmatrix} \rho\, u \\ \rho\, u^2+P-\tau_{xx} \\ \rho\, v\, u-\tau_{xy} \\ \rho\, w\, u-\tau_{xz} \\ \left(\rho(e+\dfrac{V^2}{2})u+P\,u-k\dfrac{\partial T}{\partial x}-u\,\tau_{xx}-v\,\tau_{xy}-w\,\tau_{xz}\right) \end{Bmatrix}$$

Μητρώα G,H,J

$$G = \begin{Bmatrix} \rho\, v \\ \rho\, u\, v - \tau_{yz} \\ \rho\, v^2 + P - \tau_{yy} \\ \rho\, w\, v - \tau_{yz} \\ \rho\left(e + \dfrac{V^2}{2}\right) v + P\, v - k\dfrac{\partial T}{\partial y} - u\,\tau_{yx} - v\,\tau_{yy} - w\,\tau_{yz} \end{Bmatrix} \quad H = \begin{Bmatrix} \rho\, w \\ \rho\, u\, w - \tau_{zx} \\ \rho\, v\, w - \tau_{zy} \\ \rho\, w^2 + P - \tau_{zz} \\ \rho\left(e + \dfrac{V^2}{2}\right) w + P\, w - k\dfrac{\partial T}{\partial z} - u\,\tau_{zx} - v\,\tau_{zy} - w\,\tau_{zz} \end{Bmatrix}$$

$$J = \begin{Bmatrix} 0 \\ \rho\; f_x \\ \rho\; f_y \\ \rho\; f_z \\ \rho\left(u\; f_x + v\; f_y + w\; f_z\right) + \rho\,\dot{q} \end{Bmatrix}$$

Κλείσιμο συστήματος ΔΕ

Οι προς επίλυση διαφορικές εξισώσεις διατήρησης είναι πέντε: Διατήρηση μάζας (1), διατήρηση ορμής (3), διατήρηση ενέργειας (1).

Το σύστημα των εξισώσεων διατήρησης έχει πέντε διαφορικές εξισώσεις με έξι αγνώστους (ρ, u, v, w, p, T). Η εσωτερική ενέργεια e του ρευστού εκφράζεται με μία εξίσωση της μορφής

$e = e(p,T)$ που για αέρια απλοποιείται στην $e = C_V T$ με C_V την ειδική θερμότητα του αερίου υπό σταθερό όγκο και Τα τη θερμοκρασία του αερίου.

Τέλος η πίεση, η πυκνότητα και η θερμοκρασία συνδέονται με την καταστατική εξίσωση $\rho = \rho(p,T)$
που για αέρια παίρνει τη γνωστή μορφή

$$\frac{P}{\rho} = R\,T$$

Μετασχηματισμός συντεταγμένων

Έστω καρτεσιανό σύστημα συντεταγμένων (x_1, x_2, x_3) και u_1, u_2, u_3 ένα καμπυλόγραμμο σύστημα. Μεταξύ των δύο συστημάτων υπάρχει η μονοσήμαντη σχέση μετασχηματισμού

$$x_i = x_i(u_1, u_2, u_3) \qquad (i = 1, 2, 3)$$

Ένα διαφορικό σύστημα μήκους **ds** στο καρτεσιανό σύστημα εκφράζεται $ds^2 = dx_1^2 + dx_2^2 + dx_3^2$. Ένα διαφορικό μήκος dx_i κατά μήκος του άξονα x_i εκφράζεται ως συνάρτηση των καμπυλόγραμμων ορθογωνίων συντεταγμένων

$$dx_i = \frac{\partial x_i}{\partial u_j} d\, u_j$$

Συνεπώς το διαφορικό μήκος **ds** εκφράζεται ως

$$ds^2 = \left(\frac{\partial x_1}{\partial u_j} d\, u_j \right)^2 + \left(\frac{\partial x_2}{\partial u_k} d\, u_k \right)^2 + \left(\frac{\partial x_3}{\partial u_\ell} d\, u_\ell \right)^2$$

Επειδή όμως το καμπυλόγραμμο σύστημα συντεταγμένων είναι ορθογώνιο τότε το μεικτό γινόμενο θα είναι μηδέν, προκύπτει

$$ds^2 = \left(\frac{\partial x_i}{\partial u_1}\right)^2 d\,u_1^2 + \left(\frac{\partial x_j}{\partial u_2}\right)^2 d\,u_2^2 + \left(\frac{\partial x_k}{\partial u_3}\right)^2 d\,u_3^2$$

(ι, j, k βουβοί δείκτες)

$$ds^2 = \alpha_1^2 (du_1)^2 + \alpha_2^2 (du_2)^2 + \alpha_3^2 (du_3)^2$$

όπου $$\alpha_k = \sum_{i=1}^{3} \left(\frac{\partial x_i}{\partial u_k}\right)^2 = \left(\frac{\partial x_1}{\partial u_k}\right)^2 + \left(\frac{\partial x_2}{\partial u_k}\right)^2 + \left(\frac{\partial x_3}{\partial u_k}\right)^2$$

Έκφραση ανάδελτα

Στο καμπυλόγραμμο σύστημα ορθογωνίων συντεταγμένων u_1, u_2, u_3, η διανυσματική σχέση

$$\nabla \cdot [K(x) \cdot \nabla] = \frac{1}{\alpha_1 \alpha_2 \alpha_3} \sum_{K=1}^{3} \frac{\partial}{\partial u_k} \left[\frac{\alpha_1 \alpha_2 \alpha_3}{\alpha_K^2} K(u_1 u_2 u_3) \frac{\partial}{\partial u_K} \right]$$

και για K σταθερό

$$\nabla^2 = \frac{1}{\alpha_1 \alpha_2 \alpha_3} \sum_{K=1}^{3} \frac{\partial}{\partial u_K} \left(\frac{\alpha_1 \alpha_2 \alpha_3}{\alpha_K^2} \frac{\partial}{\partial u_K} \right)$$

Κυλινδρικό σύστημα συντεταγμένων

(α) Για κυλινδρικό σύστημα συντεταγμένων

$$x = r\cos\varphi, \quad y = r\sin\varphi, \quad z = z$$

τότε

$$\alpha_r^2 = \left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2 = \cos^2\varphi + \sin^2\varphi = 1$$

$$\alpha_\varphi^2 = \left(\frac{\partial x}{\partial \varphi}\right)^2 + \left(\frac{\partial y}{\partial \varphi}\right)^2 + \left(\frac{\partial z}{\partial \varphi}\right)^2 = (-r\sin\varphi)^2 + (r\cos\varphi)^2 + 0 = r^2$$

$$\alpha_z^2 = \left(\frac{\partial x}{\partial z}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + \left(\frac{\partial z}{\partial z}\right)^2 = 0 + 0 + 1 = 1$$

Οπότε:

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}$$

(β) Για σφαιρικό σύστημα συντεταγμένων

$$x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\theta$$

$$\alpha_r = 1, \quad \alpha_\varphi = r\sin\theta, \quad \alpha_\theta = r$$

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

A general form of the equation

Cartesian velocities-Cartesian coordinates
Nearly all equations are of the form

$$\boxed{\frac{\partial}{\partial t}\left(\rho\Phi\right)+\frac{\partial}{\partial x_j}\left(\rho U_j\Phi-\Gamma_\Phi\frac{\partial\Phi}{\partial x_j}\right)=S_\Phi}$$

$$\boxed{\Phi=U_i, H, c, k, \varepsilon, \Gamma_\phi, S_\phi}$$

Characteristics

- **multidimensional**
- **turbulent flows require at least 6 equations**
- **strong coupling**
- **non-linear**

Source Terms S_{Φ}

-Ö-	-$S_{Ö}$-
1	0
u	$-\frac{\partial P}{\partial x}+\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right)+\frac{1}{r}\frac{\partial}{\partial r}\left(r\mu\frac{\partial v}{\partial x}\right)$
v	$-\frac{\partial P}{\partial r}-\frac{2\mu v}{r^2}+\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial r}\right)+\frac{1}{r}\frac{\partial}{\partial r}\left(r\mu\frac{\partial v}{\partial r}\right)$
Ô	0
k	G-ñå
å	$(C_1$åG-C_2ñå$^2)/k$
	$G=\mu\left\{2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial r}\right)^2+\left(\frac{v}{r}\right)^2\right]+\left(\frac{\partial u}{\partial r}+\frac{\partial v}{\partial x}\right)^2\right\}$

Classification of flow phenomena

Steady , unsteady
Incompressible , compressible
2D , 3D

General form of PDE

$$A\Phi_{xx}+2B\Phi_{xy}+C\Phi_{yy}+D\Phi_{x}+E\Phi_{y}+F\Phi=G$$

Where A,B,C,D,E,F and G functions of x,y ,Φ,Φ_x,Φ_y

B^2-AC=<,0,>

- **Parabolic,**
- **elliptic,**
- **hyperbolic,**
- **partially elliptic**

directional transport of information

type of boundary conditions

Model Problems

- Heat Conduction
- Boundary layers
- Recirculating flows
- Supersonic flows
- Time dependent problems

  - Boundary value problems
  - Initial value problems

Starting steps

outline of the engineering problem

select dependent variables

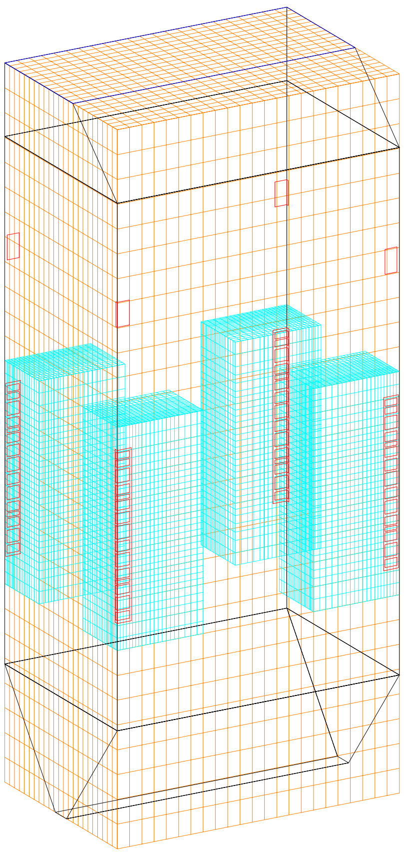
select coordinate system

write down transport equations

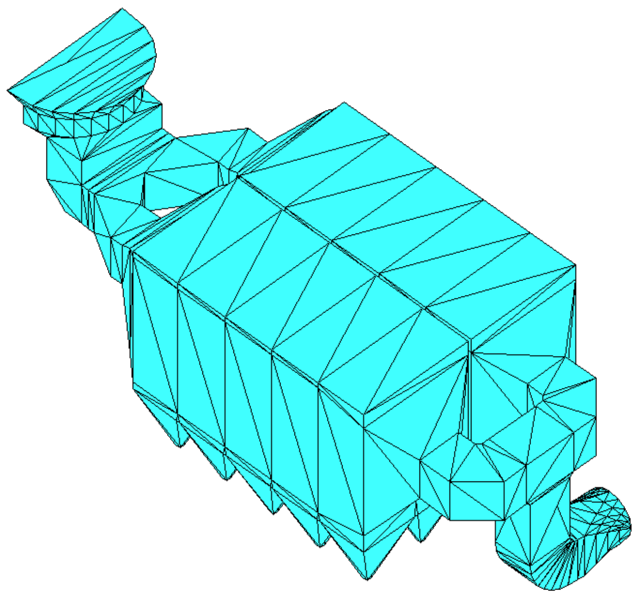
write down boundary conditions (well posed mathematical problem)

non-dimensionalize if possible

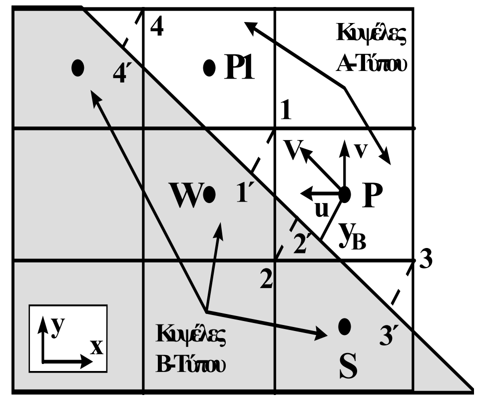
Examples of grids and dependent variables



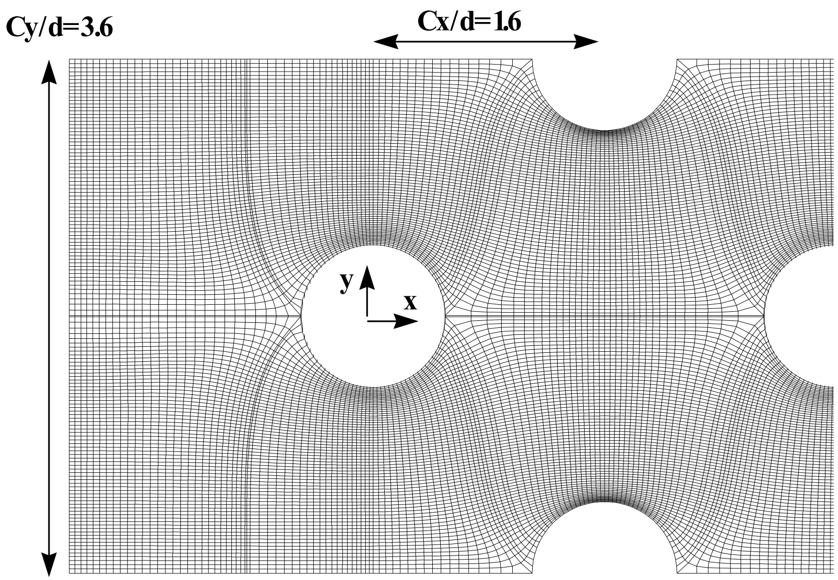
Boiler geometry

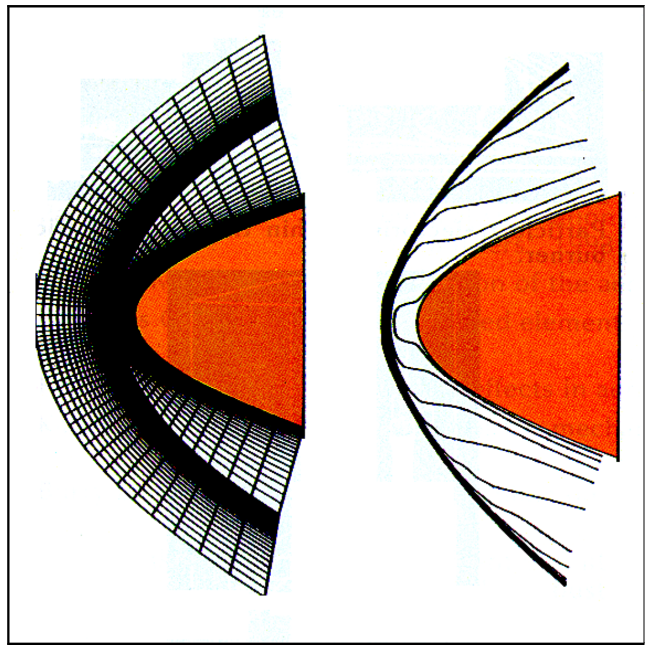


E/P geometry



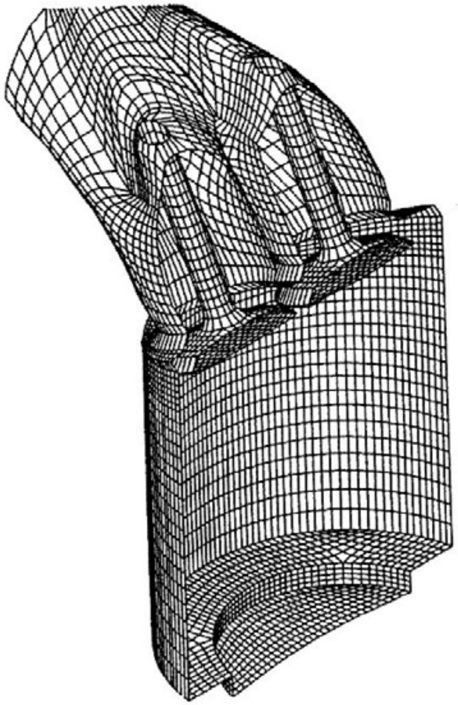
Examples of grid/2



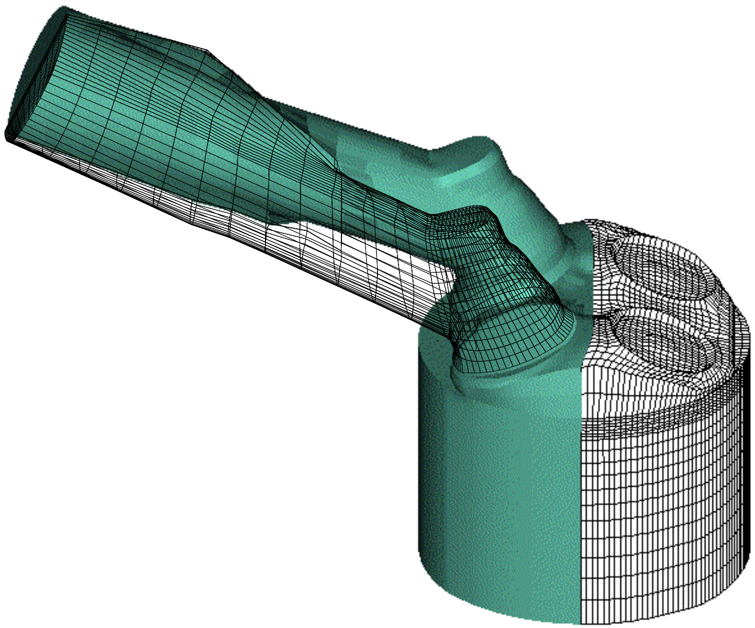


Examples of grid/3

The intake port (UP Valencia)



An I.C engine



A graphics package is a must

Solution of a Poisson equation with Dirichlet or Neumann boundary conditions-forcing functions

"A Laplacian Equation Method for Numerical Generation of Boundary-fitted 3D Orthogonal Grids"
T.Theodoropoulos, G. C. Bergeles
Journal of Computational Physics, vol. 82,No 2,1989,pp269-288

"Numerical Grid Generation Technique for 3D Complex Spaces"
Glekas, J., Bergeles, G., Athanassiades, N.
3rd Intern. Conference, Computational Methods and Experimental measurements, Sept 1986,Porto Carras,Springer Verlag,Vol 2,pp905-916